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Chaotic maps on measure spaces and behavior of states

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Introduction. As well known, chaotic maps are considered as those φ's which have the following property (cf.[1]).

(1) The set of all periodic points for φ are dense.

(2) φ is transitive.

(3) φ depends on sensitive initial condition.

Those properties are concerned with the orbit of a given initial point. In this note, we consider how probability density functions changed by iteration of chaotic maps. More generally, we study behavior of states by $*$-endomorphisms of von Neumann algebras associated with chaotic maps. In particular, we show some theorems concerning the limits of iterated states ,which are stated as follows.

(4) The sequence of iterated states by a chaotic map converges to a unique state in the norm topology.

In Section 1 and 2, we note some results related to $*$-endomorphisms of von Neumann algebras and iterated states by chaotic maps respectively, which are stated without proof. Section 3 consists of examples only which give us the meaning of theorems in Section 2 and provide fruitful discussion on our theory. Moreover we can find deep relationship between our study and wavelets theory (cf.[4]). This note is a continuation of [5].

§1. A $*$-endomorphism of von Neumann algebra associated with a family of isometries. Let $\mathcal{H}$ be a Hilbert space with inner product $< \cdot, \cdot >$. In this note $\{V_i\}_{i=1}^n$ means a family of isometries on $\mathcal{H}$ satisfying the following property and is said to be a FIC on $\mathcal{H}$ for short.

(C.1) $\{V_iV_i^*\}_{i=1}^n$ is a set of mutually orthogonal projections and $\sum_{i=1}^n V_iV_i^* = I$.

Of course, this family $\{V_i\}_{i=1}^n$ on $\mathcal{H}$ is the generators of the image of a representation of Cuntz-algebra $\mathcal{O}_n$ [3]. Moreover we can define a $*$-endomorphism α_V of the full operator algebra $B(\mathcal{H})$ as follows.

(C.2) $\alpha_V(T) = \sum_{i=1}^n V_iTV_i^*, (T \in B(\mathcal{H}))$

If a von Neumann algebra M on $\mathcal{H}$ is invariant for α_V, then α_V becomes a $*$-endomorphism of M. For n and a positive integer k, we denote by $I(n)$ the set $\{1, 2, \ldots n\}$ and $I(n)^k$ the set of all k-tuples $\mu = (j_1, \ldots, j_k)$ with j_i in $\{1, 2, \ldots n\}$. For μ in $I(n)^k$ we denote by $V(\mu)$ the isometry $V_{j_1} V_{j_2} \cdots V_{j_k}$ on $(\mathcal{H})$. Then $\{V(\mu) | \mu \in I(n)^k\}$ is a family of isometrics whose final projections are mutually orthogonal. When α_V is a $*$-endomorphism of M, α_V^n is of the form:

$$\alpha_V^k(T) = \sum_{\mu \in I(n)^k} V(\mu) T V(\mu)^*, \quad (T \in M).$$

Proposition 1.1. *Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$ and e a unit vector in $\mathcal{H}$ such that $V_1 e = e$. We put*

$$ONS(e, V) = \bigcup_{k=1}^{\infty} \{V(\mu) e | \mu \in I(n)^k\}.$$

Then $ONS(e, V)$ is an orthonormal system.

Remark. An orthonormal system $ONS(e, V)$ in the proposition above is regarded as the sequence $\{e_k\}_{k=1}^{\infty}$ which is inductively defined as follows: $e_1 = e$ and

$$e_{i+n(\ell-1)} = V_i e_\ell \quad (i \in I(n), \ell \in \mathbf{N}).$$

(c.f. 2 of [2])

For a von Neumann algebra M on $\mathcal{H}$, M_* denotes the predual of M. We denote by α_V^* the transpose map of α_V with respect to the duality of M and M_*. The vector state in M_* associated with unit vector ξ in $\mathcal{H}$ is denoted by ω_ξ, that is, for T in M, $\omega_\xi(T) = < T\xi, \xi >$ and

$$\omega_\xi(\alpha_V(T)) = < \alpha_V(T)\xi, \xi > = \alpha_V^*(\omega_\xi)(T).$$

Moreover we have

$$\alpha_V^*(\omega_\xi) = \sum_{i=1}^n \omega_{V_i^* \xi}.$$

When e is a unit vector such that $V_1 e = e$, namely, it is an eigenvector for eigenvalue 1 of V_1, we denote by $\mathcal{H}_e$ the subspace of $\mathcal{H}$ spanned by $ONS(e, V)$.

Proposition 1.2. *Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$. If there exists a unit vector e such that $V_1 e = e$, then for any unit vector ξ in the subspace $\mathcal{H}_e$ it follows that*

$$\lim_{n \to} (\alpha_V^*)^n(\omega_\xi) = \omega_e \quad (\textit{norm topology}).$$

Proposition 1.3. *Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$. If there exists a unit vector e such that $V_1 e = e$, then for any state ω of the form $\omega = \sum_{k=1}^{\infty} \omega_{\xi_k}$ where ξ_k's are in $\mathcal{H}_e$, it follows* *that*

$$\lim_{n\to}(\alpha_V^*)^n(\omega) = \omega_e \quad (\textit{norm topology}).$$

Proposition 1.4. *Let $\{V_i\}_{i=1}^n$ be a FIC on $\mathcal{H}$ and e a unit vector such that $V_1 e = e$, If $ONS(e, V)$ is complete, then for any state ω in the predual of $B(\mathcal{H})$ it follows that*

$$\lim_{n\to\infty}(\alpha_V^*)^n(\omega) = \omega_e \quad (\textit{norm topology}).$$

Proposition 1.5. *Let M be a Neumann algebra on $\mathcal{H}$ and $\{V_i\}_{i=1}^n$ and $\{W_i\}_{i=1}^n$be a couple of families of isometries on $\mathcal{H}$ satisfying* (1.1). *Suppose that M is invariant for α_V and α_W. Then following conditions are equivalent.*

(1) $\alpha_V(T) = \alpha_W(T)$ *for all T in M.*

(2) $$(W_1.\cdots, W_n) = (V_1.\cdots, V_n)\begin{pmatrix} h_{11} & \cdots & h_{1n} \\ \vdots & \ddots & \vdots \\ h_{n1} & \cdots & h_{nn} \end{pmatrix},$$

that is, $W_i = \sum_{j=1}^{n} V_j h_{ji}$, $(1 \le i \le n)$, where each h_{ij} is a unitary element in the commutant M' of M on the Hilbert space $\mathcal{H}$.

§**2. Chaotic maps and behavior of states.** Let X be a measure space with measure m and φ a measurable map on X, Here we note some notations concerning X and φ.

(1) $m \circ \varphi$ denotes the measure on X defined by $m \circ \varphi(E) = m(\varphi(E))$ and if the map φ is absolutely continuous with respect to m, the Radon-Nikodym derivative for $m \circ \varphi$ and m is denoted by $\dfrac{dm \circ \varphi}{dm}$

(2) α_φ denotes the $*$-endomorphism of $L^\infty(X) = L^\infty(X, m)$ defined by $\alpha_\varphi(f) = f(\varphi(x))$ for f in $L^\infty(X)$.

(3) T_φ denotes the linear operator on the Hilbert space $\mathcal{H}$=$L^2(X) = L^2(X, m)$ defined by $(T_\varphi \xi)(x) = \xi(\varphi(x))$ for ξ in $\mathcal{H}$.

(4) For a subset Y of X, χ_Y means the characteristic function of Y.

(5) For a measurable function f on X, M_f denotes the multiplication operator on $L^2(X)$ defined by $M_f \xi = f\xi$ for ξ in $L^2(X)$.

(6) For f in $L^\infty(X)$, $\pi(f)$ denotes the bounded multiplication operator on $L^2(X)$ defined by $\pi(f)\xi = f\xi$ for ξ in $L^2(X)$.

Definition 2.1. Let X is a measure space with measure m. A measurable map φ of X onto X is said to be a map with n-laps , MWnL for short, if there exists n measurable subsets $\{X_i\}_{i=1}^n$ of X such that

(1) $\cup_{i=1}^n X_i = X$ and $X_i \cap X_j = \phi$ for $i \neq j$.

(2) Each restriction φ_i of φ to X_i is a bimeasurable map of X_i onto X in the sense that φ_i is an surjective map of X_i onto $\varphi_i(X_i)$ with $m(X \backslash \varphi_i(X_i)) = 0$ and φ_i^{-1} is measurable, too.

(3) For each i, φ_i and φ_i^{-1} are absolutely continuous with respect to m and non-singular in the sense that

$$\frac{dm \circ \varphi}{dm}(x) \neq 0, \ a.e.x \quad \text{and} \quad \frac{dm \circ \varphi^{-1}}{dm}(x) \neq 0, \ a.e.x.$$

For a measure space (X, m) and a measurable map φ of X into itself, M_f and T_φ is not necessarily defined on the full space $\mathcal{H}$. Then each isometry V_i in the following definition, if necessary, is considered as a uniquely extended bounded linear operator on the full Hilbert space $\mathcal{H}$.

Definition 2.2. Let φ be a MWnL on a measure space (X, m). We define a family isometries $\{V_i(\varphi)\}_{i=1}^n$ associated with φ as follows.

$$V_i(\varphi) = M_{\sqrt{dm\circ\varphi/dm}} M_{\chi_{X_i}} T_\varphi \quad (i = 1, \ldots n),$$

By the definition we can see that

(1) $V_i(\varphi)^* = M_{\sqrt{dm\circ\varphi_i^{-1}/dm}} T_{\varphi_i^{-1}} \quad (i = 1, \ldots n)$.

(2) $V_i(\varphi)V_i(\varphi)^* = M_{\chi_{X_i}} \quad (i = 1, \ldots n)$.

(3) $\displaystyle\int_X f(\varphi(x))\eta(x)dm(x) = \sum_{i=1}^n \int_X \frac{dm \circ \varphi_i^{-1}}{dm} \eta(\varphi_i^{-1}(x))dm$ for η in $L^1(X, m)$.

Proposition 2.3. *Let φ be a MWnL on a measure space (X, m) and $\{V_i = V_i(\varphi)\}_{i=1}^n$ a family isometries associated with φ defined in Definition* 2.2. *Then it follows that*

(1) $\{V_i\}_{i=1}^n$ *satisfies condition* (C.1) *in* §1, *that is,* $\{V_i\}_{i=1}^n$ *is a FIC on* $L^2(X, m)$.

(2) $\pi(\alpha_\varphi(f)) = \alpha_V(\pi(f))$ *for all* f *in* $L^\infty(X)$.

Proposition 2.3 (2) implies that α_V is a $*$-endomorphism of the von Neumann algebra $M_{L^\infty(X)}$ and we denote by A_φ the transpose of the restriction of α_V to $M_{L^\infty(X)}$. Then we have

$$(A_\varphi \eta)(x) = \sum_{i=1}^{n} \frac{dm \circ \varphi_i^{-1}}{dm} \eta(\varphi_i^{-1}(x)).$$

The transformation A_φ is known as Perron-Frobenius operator on $L^1(X, m)$.

Theorem 2.4. *Let φ be a MWnL on a measure space (X, m). Suppose that there exists a FIC $\{W_i\}_{i-1}^n$ such that W_1 has eigenvalue 1 with eigenvector e and*

$$\alpha_V(T) = \alpha_W(T) \quad \textit{for } T \textit{ in } M,$$

where M is a von Neumann algebra on $\mathcal{H}$. Then for any state ω of the form $\omega = \sum_{k=1}^{\infty} \omega_{\xi_k}$ where ξ_k's are in $\mathcal{H}_e$, it follows that

$$\lim_{n\to\infty} (\alpha_V^*)^n(\omega) = \omega_e \quad (\textit{norm topology on } M_*).$$

Moreover, this implies that

$$\lim_{n\to\infty} \|A_\varphi^n(\eta) - |e|^2\|_1 = 0.$$

where $\eta = |\xi|^2$ for ξ in $\mathcal{H}_e$.

Proposition 2.5. *Let φ be a MW2L on the interval $[0, 1]$ with Lebesgue measure m. Then the following conditions are equivalent.*

(1) *$V_1(\varphi)$ has eigenvalue 1 with eigenvector e.*

(2) $m(\{x \in [0,1] | \frac{d\circ\varphi_1}{dm}(x) = 1\}) > 0$.

Theorem 2.6. *Let φ be a MWnL on a measure space (X, m) and $e(x) = 1$ for a.e. x in X. Then following conditions are equivalent.*

(1) *There exists a FIC $\{W_i\}_{i-1}^n$ such that $\alpha_V(T) = \alpha_W(T)$ for T in $M_{L^\infty(X)}$ and $W_1 e = e$.*

(2) *T_φ is an isometry.*

(3) $\displaystyle\sum_{i=1}^{n} \frac{dm \circ \varphi_i^{-1}}{dm}(x) = 1$ *for a.e. x in X.*

Definition 2.7. Let φ and ψ be two MWnL's on (X, m). Two maps are said to be AC-topologically conjugate if there exists a bijective map h of X onto itself satisfying following conditions.

(1) $\varphi = h \circ \psi \circ h^{-1}$.

(2) Both $m \circ h$ and $m \circ h^{-1}$ are absolutely continuous and non-singular with respect to m.

Remark. Let h be a absolutely continuous map satisfying (2) of the definition above. We put

$$U(h) = M_{\sqrt{dm \circ h/dm}} T_h.$$

Then $U(h)$ is a unitary operator on $\mathcal{H}$.

Theorem 2.8. *Let φ and ψ be two MWnL's on (X, m). Suppose that ψ is AC-conjugate to φ and there exists a FIC $\{W_i\}_{i=1}^n$ satisfying following conditions.*
(1) W_1 has eigenvalue 1 with unit eigenvector e.
(2) $\alpha_{V(\varphi)}(T) = \alpha_W(T)$ for T in M,
where M is a von Neumann algebra on $\mathcal{H}$. Let $f = U(h^{-1})e$. Then for any state ω of the form $\omega = \sum_{k=1}^{\infty} \omega_{\xi_k}$ where ξ_k's are in $\mathcal{H}_f$, it follows that

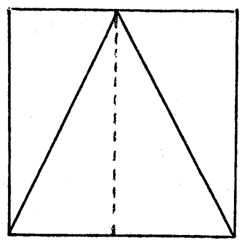
$$\lim_{n \to \infty} (\alpha_V^*)^n(\omega) = \omega_e \quad (\textit{norm topology on } (U(h)MU(h)^*)_*).$$

§3. **Examples of MWnL.** We give typical and interesting examples of map with n laps. Each number in each example indicates the following.
(1) Measure space (X,m) on which a map is given.
(2) Map φ with n laps on X.
(3) Number n and partition $\{X_i\}_{i=1}^n$ of X.
(4) $\{V_i\}_{i=1}^n = \{V_i(\varphi)\}_{i=1}^n$ defined in Definition 2.2.
(4-1) An eigenvector e for eigenvalue 1 of W_1 and $ONS(e, V) = \{e_k\}_{k=1}^{\infty}$.
(4-2) $ONS(e, V)$ is complete or not.
(5) $\{W_i\}_{i=1}^n$ such that $\alpha_V(T) = \alpha_W(T)$ for T in a von Neumann algebra M on $L^2(X, m)$.
(6) The von Neumann algebra M on which $\alpha_V = \alpha_W$.
(6-1) An eigenvector e for eigenvalue 1 of W_1 and $ONS(e, W) = \{e_k\}_{k=1}^{\infty}$.
(6-2) $ONS(e, W)$ is complete or not.
(7) Perron-Frobenius operator A_φ.

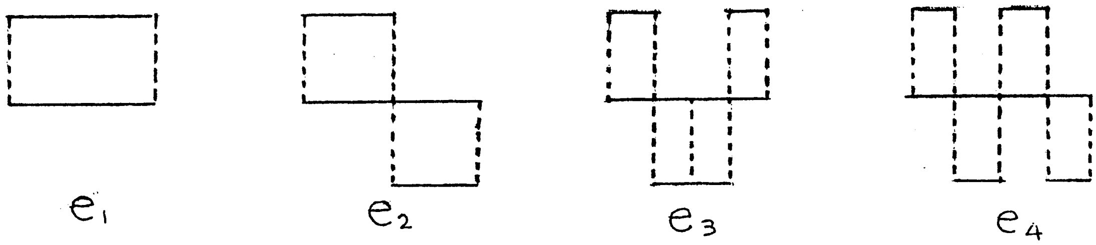
Example 3.1. (Tent map)
(1) $X = [0, 1]$, and m =Lebesgue measure.
(2) φ is the map τ defined by

$\tau(x) = 1 - |1 - 2x|$.

(3) $n = 2$ and $X_1 = [0, 1/2), X_2 = [1/2, 1]$.
(4) $V_1 = \sqrt{2} M_{[0,1/2)} T_\tau, \quad V_1 = \sqrt{2} M_{[1/2,1]} T_\tau$.
(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$.

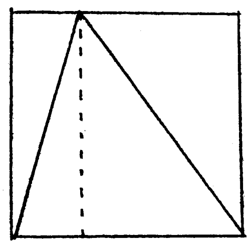


(6) $M = B(L^2[0,1])$
(6-1) $e(x) = 1$ $(x \in [0,1])$ and $e_1 = e$, $e_2 = M_{[0,1/2)}e_1 - M_{[1/2,1]}e_1$.
(6-2) $ONS(e, W)$ is complete.
(7) $A_\tau(\eta)(x) = \frac{1}{2}\left(\eta\left(\frac{x}{2}\right) + \eta\left(1 - \frac{x}{2}\right)\right)$.

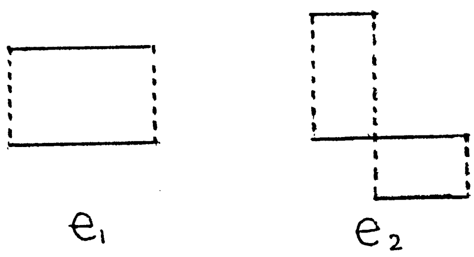


Example 3.2. (Generalized tent map)
(1) $X = [0,1]$, and m =Lebesgue measure.
(2) $\varphi = \tau_c$, $(0 < c < 1)$ defined by

$$\varphi_c(x) = \begin{cases} \frac{1}{c}x & \text{for } 0 \le x \le c, \\ \frac{1}{c-1}(x-1) & \text{for } c < x \le 1. \end{cases}$$

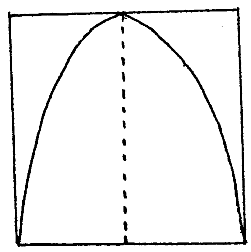


(3) $n = 2$ and $X_1 = [0,1/2), X_2 = [1/2,1]$.
(4) $V_1 = M_{\sqrt{1/c}} M_{\chi_{[0,c]}} T_{\tau_c}$, $\quad V_2 = M_{\sqrt{1/(1-c)}} M_{\chi_{[c,1]}} T_{\tau_c}$.
(5) $(W_1, W_2) = (V_1, V_2)\begin{pmatrix} \sqrt{c} & \sqrt{1-c} \\ \sqrt{1-c} & -\sqrt{c} \end{pmatrix}$.
(6) $M = B(L^2[0,1])$
(6-1) $e(x) = 1$ $(x \in [0,1])$ and $e_1 = e$, $e_2(x) = \begin{cases} \frac{1}{\sqrt{c}} & \text{for } 0 \le x \le c, \\ -\frac{1}{\sqrt{c-1}} & \text{for } c < x \le 1. \end{cases}$
(6-2) $ONS(e, W)$ is complete.
(7) $A_{\tau_c}(\eta)(x) = c(\eta(cx)) + (1-c)\eta((c-1)x+1))$.



Remark. τ_c and τ_c are topologically conjugate (cf.[6],[8]) but they are AC-conjugate only if $c = d$.

Example 3.3.(Logistic map) (cf.[9])
(1) $X = [0,1]$, and m =Lebesgue measure.
(2) φ is the map λ defined by
$\lambda(x) = 4x(1-x)$ (3) $n = 2$ and $X_1 = [0,1/2), X_2 = [1/2,1]$.



(4)$V_1 = \dfrac{1}{2\sqrt{1-2x}} M_{\chi_{[0,1/2]}} T_\lambda, \quad V_2 = \dfrac{1}{2\sqrt{2x-1}} M_{\chi_{[1/2,1]}} T_\lambda.$

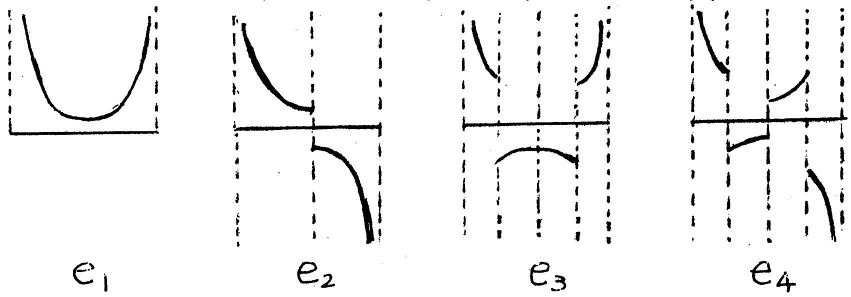
(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$

(6) $M = B(L^2[0,1])$

(6-1) $e_1(x) = e(x) = 1/\sqrt{\pi\sqrt{x(1-x)}}$ and $e_2(x) = M_{\chi_{[0,1/2)}} e_1 - M_{\chi_{[1/2,1]}} e_1.$

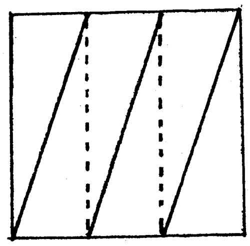
(6-2) $ONS(e, W)$ is complete.

(7) $A_{\tau_c}(\eta)(x) = \dfrac{1}{4\sqrt{1-x}} \left(\eta\left(\dfrac{1-\sqrt{1-x}}{2}\right) - \eta\left(\dfrac{1+\sqrt{1-x}}{2}\right)\right).$



(The logistic map is topologically conjugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$ (cf.[7])).

Example 3.4. (Typical map with 3 laps)



(1) $X = [0,1]$, and m =Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x) = \begin{cases} 3x & \text{for } 0 \le x < 1/3, \\ 3x - 1 & \text{for } 1/3 \le x < 2/3, \\ 3x - 2 & \text{for } 2/3 \le x \le 1. \end{cases}$$

(3) $n = 3$ and $X_1 = [0, 1/3), X_2 = [1/3, 2/3), X_2 = [2/3, 1]$.

(4) $V_1 = \sqrt{3} M_{\chi_{[0,1/3)}} T_\varphi, \quad V_2 = \sqrt{3} M_{\chi_{[1/3,2/3)}} T_\varphi, \quad V_3 = \sqrt{3} M_{\chi_{[2/3,1]}} T_\varphi.$

(5) $(W_1, W_2, W_3) = (V_1, V_2, V_3) \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & (3-\sqrt{3})/6 & (-3-\sqrt{3})/6 \\ 1/\sqrt{3} & (-3-\sqrt{3})/6 & (3-\sqrt{3})/6 \end{pmatrix}.$

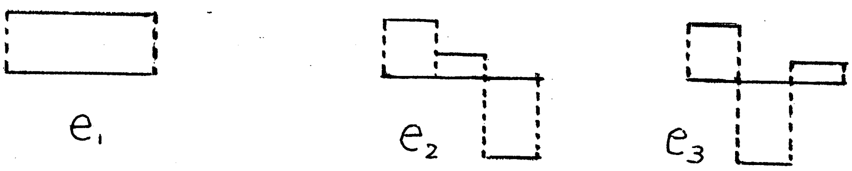
(6) $M = B(L^2[0,1])$

(6-1) $e(x) = 1 \ (x \in [0,1])$ and $e_1 = e$, $e_2(x) = \chi_{[0,1/3)} + \dfrac{\sqrt{3}-1}{2}\chi_{[1/3,2/3)} + \dfrac{-\sqrt{3}-1}{2}\chi_{[2/3,1]}$,

$e_3(x) = \chi_{[0,1/3)} + \dfrac{-\sqrt{3}-1}{2}\chi_{[1/3,2/3)} + \dfrac{\sqrt{3}-1}{2}\chi_{[2/3,1]}.$

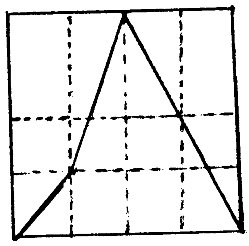
(6-2) $ONS(e, W)$ is complete.

(7) $A_\varphi(\eta)(x) = \dfrac{1}{3}\left(\eta\left(\dfrac{x}{3}\right) + \eta\left(\dfrac{x}{3} + \dfrac{1}{3}\right) + \eta\left(\dfrac{x}{3} + \dfrac{2}{3}\right)\right).$



Example 3.5. (MW2L on $[0,1]$ such that V_1 has an eigenvector for eigenvalue 1:a)

(1) $X = [0,1]$ and m is the Lebesgue measure.



(2) φ is the map defined by

$$\varphi(x) = \begin{cases} x & \text{for } 0 \le x < 1/4, \\ (6x-1)/2 & \text{for } 1/4 \le x < 1/2, \\ -2x+2 & \text{for } 1/2 \le x \le 1 \end{cases}$$

(3) $n = 2$ and $X_1 = [0,1/2), X_2 = [1/2,1]$

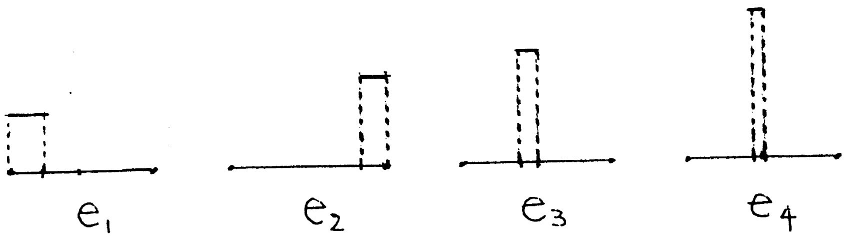
(4)$V_1 = M_{\chi_{[0,1/4)}} + \sqrt{3}M_{\chi_{[1/4,1/2)}}$, $V_2 = \sqrt{2}M_{\chi_{[1/2,1]}}$.

(4-1) $e_1 = e = 2\chi_{[0,1/4)}$, $e_2 = 2\sqrt{2}\chi_{(7/8,1]}$, $e_3 = 2\sqrt{6}\chi_{(11/24,1/2]}$ $e_4 = 4\chi_{[1/2,9/16)}$.

(4-2) $ONS(e,V)$ is not complete.

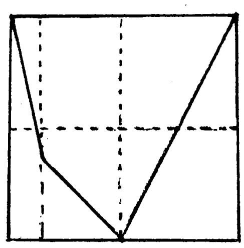
(6) $M = B(L^2[0,1])$

(7) $A_\varphi(\eta)(x) = \eta(x)\chi_{[0,1/4]}(x) + \frac{1}{\sqrt{3}}\eta\left(\frac{2x+1}{6}\right)\chi_{[1/4,1]} + \frac{1}{\sqrt{2}}\eta\left(\frac{-x+2}{2}\right)$.



Example 3.6. (MW2L on $[0,1]$ such that V_1 has an eigenvector for eigenvalue 1:b)

(1) $X = [0,1]$ and m is the Lebesgue measure.



(2) φ is the map defined by

$$\varphi(x) = \begin{cases} -5x+1 & \text{for } 0 \le x < 1/8, \\ -x+(1/2) & \text{for } 1/8 \le x < 1/2, \\ 2x-1 & \text{for } 1/2 \le x \le 1 \end{cases}$$

(3) $n = 2$ and $X_1 = [0,1/2)$, $X_2 = [1/2,1]$

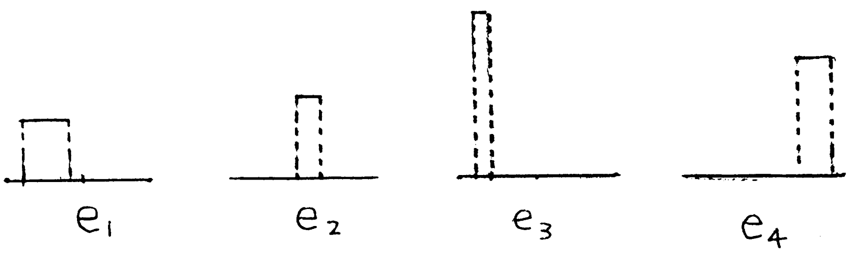
(4)$V_1 = \sqrt{5}M_{\chi_{[0,1/8)}} + M_{\chi_{[1/8,1/2)}}$, $V_2 = \sqrt{2}M_{\chi_{[1/2,1]}}$.

(4-1) $e_1 = e = 2\chi_{[1/8,3/8]}$, $e_2 = 2\sqrt{2}\chi_{[9/16,11/16]}$, $e_3 = 2\sqrt{10}\chi_{[5/80,7/80]}$ $e_4 = 4\chi_{[25/32,27/32]}$.

(4-2) $ONS(e,V)$ is not complete.

(6) $M = B(L^2[0,1])$

(7) $A_\varphi(\eta)(x) = \eta\left(\frac{-2x+1}{2}\right)\chi_{[0,1/8)}(x) + \frac{1}{\sqrt{5}}\eta\left(\frac{-x+1}{5}\right)\chi_{[1/8,1]} + \frac{1}{\sqrt{2}}\eta\left(\frac{x+1}{2}\right)$.

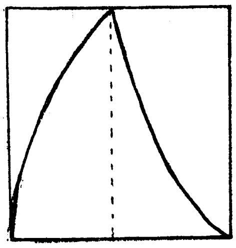


Example 3.7.(Square root map)

(1) $X=[0,1]$ and m =Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x)=\begin{cases}\sqrt{2x} & \text{for } 0\le x<1/2,\\ 1-\sqrt{2x-1} & \text{for } 1/2\le x\le 1.\end{cases}$$



(3) $n=2$ and $X_1=[0,1/2),\ X_2=[1/2,1]$.

(4)$V_1=(1/\sqrt{2x})M_{\chi_{[0,1/2)}}T_\varphi,\ V_2=(1/\sqrt{2x-1})M_{\chi_{[1/2,1]}}T_\varphi$.

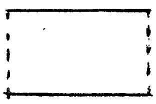
(5)$(W_1,W_2)=(V_1,V_2)\begin{pmatrix} M_{\sqrt{2cx}} & M_{\sqrt{1-2cx}}\\ M_{\sqrt{1-2cx}} & M_{\sqrt{2cx}}\end{pmatrix}$.

(6) $M=M_{L^\infty[0,1]}$

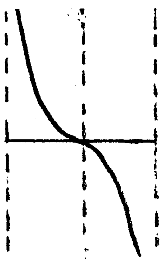
(6-1) $e_1(x)=e(x)=1,\ e_2(x)=\sqrt{(1/\sqrt{2x})-1}\chi_{[0,1/2)}(x)-\sqrt{(1/\sqrt{2x-1})-1}\chi_{[1/2,1]}(x)$

(6-2) Now we cannot find whether $ONS(e,W)$ is complete or not.

(7)$A_\varphi(\eta)(x)=\frac{1}{x}\left(\eta\left(\frac{x^2}{2}\right)+\frac{1}{x-1}\eta\left(\frac{x^2-2x+2}{2}\right)\right)$.



e_1



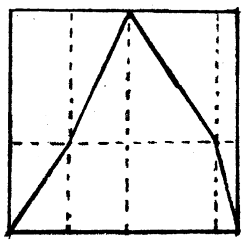
e_2

Example 3.8.(Map of broken line)

(1) $X=[0,1]$ and m =Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x)=\begin{cases} 8x/5 & \text{for } 0\le x<1/4,\\ (12x-1)/5 & \text{for } 1/4\le x<1/2,\\ (-12x+13)/7 & \text{for } 1/2\le x<17/20,\\ (-8x+8)/3 & \text{for } 7/20\le x\le 1,\end{cases}$$



(3) $n=2$ and $X_1=[0,1/2),\ X_2=[1/2,1]$.

(4)$V_1=(\sqrt{8/5}M_{\chi_{[0,1/4)}}+\sqrt{12/5}M_{\chi_{[1/4,1/2)}})T_\varphi,\quad V_2=(\sqrt{12/7}M_{\chi_{[1/2,17/20)}}+\sqrt{8/3}M_{\chi_{[17/20,1]}})T_\varphi$

(5)$(W_1,W_2)=(V_1,V_2)\begin{pmatrix}\sqrt{5/8}M_{\chi_{[0,2/5)}}+\sqrt{5/12}M_{\chi_{[2/5,1]}}, & \sqrt{3/8}M_{\chi_{[0,2/5)}}+\sqrt{7/12}M_{\chi_{[2/5,1]}}\\ \sqrt{3/8}M_{\chi_{[0,2/5)}}+\sqrt{7/12}M_{\chi_{[2/5,1]}}, & \sqrt{5/8}M_{\chi_{[0,2/5)}}-\sqrt{5/12}M_{\chi_{[2/5,1]}}\end{pmatrix}$.

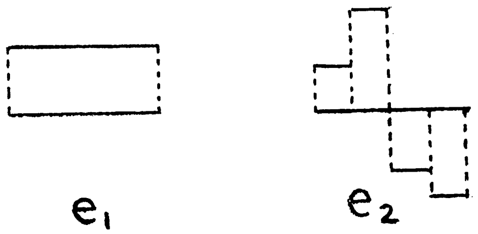
(6) $M=B(L^2[0,2/5])\oplus B(L^2[2/5,1])$

(6-1) $e_1(x)=e(x)=1,\ e_2(x)=\begin{cases}\sqrt{3/5} & \text{for } 0\le x<1/4,\\ \sqrt{7/5} & \text{for } 1/4\le x<1/2,\\ -\sqrt{5/7} & \text{for } 1/2\le x<17/20,\\ -\sqrt{5/3} & \text{for } 7/20\le x\le 1,\end{cases}$

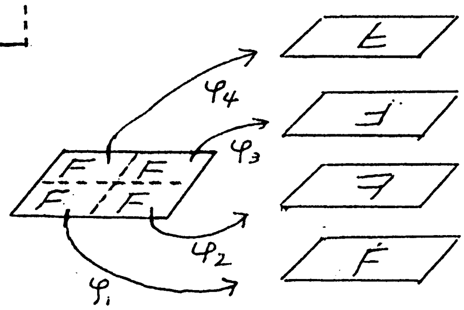
(6-2) Now we cannot find whether $ONS(e, W)$ is complete or not.

(7) $A_\varphi(\eta)(x) = \frac{5}{8}\eta\left(\frac{5x}{8}\right)\chi_{[0,2/5)}(x) + \frac{5}{12}\eta\left(\frac{5x+1}{12}\right)\chi_{[2/5,1]}(x)$

$$+\frac{3}{8}\eta\left(\frac{-3x+8}{8}\right)\chi_{[0,2/5)}(x) + \frac{7}{12}\eta\left(\frac{-7x+13}{12}\right)\chi_{[2/5,1]}(x).$$



Example 3.9. (Product of tent maps)



(1) $X = [0,1] \times [0,1]$ and m =Lebesgue measure.

(2) φ is the map defined by

$$\varphi(x,y) = (\tau(x), \tau(y))$$

where τ is the tent maps defined in Example 3.1.

(3) $n = 4$ and $X_1 = [0,1/2)\times[0,1/2)$, $X_2 = [1/2,1]\times[0,1/2)$, $X_3 = [1/2,1]\times[1/2,1]$, $X_4 = [0,1/2)\times[1/2,1]$.

(4)$V_1 = 2M_{\chi_{[0,1/2)\times[0,1/2)}}T_\varphi$, $V_2 = 2M_{\chi_{[1/2,1]\times[0,1/2)}}T_\varphi$, $V_3 = 2M_{\chi_{[0,1/2]\times[1/2,1]}}T_\varphi$, $V_4 = 2M_{\chi_{[1/2,1]\times[1/2,1]}}T_\varphi$.

(5) $$(W_1, W_2, W_3, W_4) = (V_1, V_2, V_3, V_4)\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{pmatrix}$$

(6) $M = B(L^2([0,1]\times[0,1]))$

(6-1) $e_1(x,y) = e(x,y) = 1$ $((x,y) \in [0,1]\times[0,1])$ and

$e_2(x) = \chi_{[0,1/2)\times[0,1/2)} - \chi_{[1/2,1]\times[0,1/2)} + \chi_{[0,1/2]\times[1/2,1]} - \chi_{[1/2,1]\times[1/2,1]}$.

(6-2) $ONS(e, W)$ is complete.

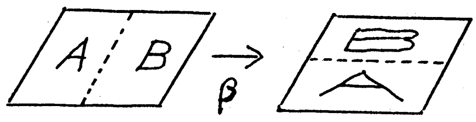
(7) $A_\varphi(\eta)(x) = \frac{1}{4}\left(\eta\left(\frac{x}{2},\frac{x}{2}\right) + \eta\left(1-\frac{x}{2},\frac{x}{2}\right) + \eta\left(\frac{x}{2},1-\frac{x}{2}\right) + \eta\left(1-\frac{x}{2},1-\frac{x}{2}\right)\right)$.

Example 3.10. (Baker's transformation)

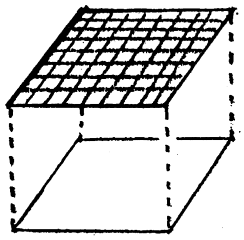
(1) $X = [0,1]\times[0,1]$ and m =Lebesgue measure.

(2) φ is the map β defined by

$$\beta(x,y) = \begin{cases} (2x, y/2) & \text{for } 0 \le x < 1/2, \\ (2x-1, (y+1)/2) & \text{for } 1/2 \le x \le 1. \end{cases}$$



(3) $n = 1$ and $X_1 = X$
(4) $V_1 = T_\beta$
(4-1) $e_1(x,y) = e(x,y) = 1$
(4-2) $ONS(e, W) = \{e_1\}$ is not complete.
(6) $M = B(L^2([0,1] \times [0,1]))$
(7) $A_\beta(\eta)(x) = \eta(\beta(x))$



e_1

Remark. Baker's transformation is strong-mixing but $\{(\alpha_V^*)^n(\omega_\xi)\}_{n=1}^\infty$ does not converges to ω_e in the norm topology in M_*.

Example 3.11.(Unilateral shift map)
(1) $X = \prod_{n=1}^\infty \{1,2\}$ and m =usual measure.
(2)φ is the map σ defined by

$$\sigma((x_1, x_2, x_3, \ldots)) = (x_2, x_3, x_4, \ldots),$$

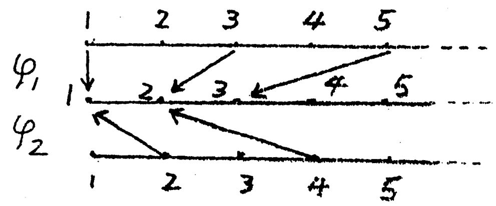
(3) $n = 2$ and $X_1 = X(1) = \{(x_n)_{n=1}^\infty \in X | x_1 = 1\}, X_2 = X(2) = \{(x_n)_{n=1}^\infty \in X | x_1 = 2\}$
(4) $V_1 = \sqrt{2} M_{X(1)} T_\sigma, \quad V_1 = \sqrt{2} M_{X(2)} T_\sigma.$

(5) $(W_1, W_2) = (V_1, V_2) \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$
(6) $M = B(L^2(X))$
(6-1) $e(x) = 1$ $(x \in X)$ and $e_1 = e$, $e_2 = \chi_{X(1)} e_1 - \chi_{X(2)} e_1$.
(6-2) $ONS(e, W)$ is complete.
(7) $A_\sigma(\eta)(x) = \dfrac{1}{2}(\eta(\gamma_1) + \eta(\gamma_2))$,
where $\gamma_1((x_1, x_2, x_3, \ldots)) = (1, x_1, x_2, \ldots)$ and $\gamma_2((x_1, x_2, x_3, \ldots)) = (2, x_1, x_2, \ldots)$.

Example 3.12. (MW2L on the set N of all natural numbers)
(1) $X =$ N and m is the counting measure.
(2) φ is the map defined by

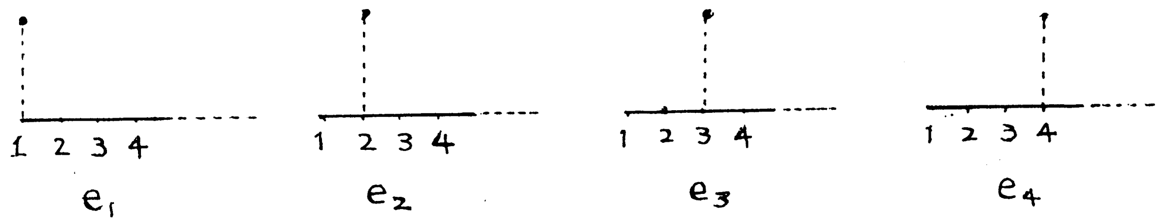
$$\varphi(2k-1) = k \text{ and } \varphi(2k) = k \ (k \in \mathrm{N})$$

(3) $n = 2$ and $X_1 = 2\,\mathrm{N} - 1$, $X_2 = 2\,\mathrm{N}$.
(4)$V_1 = M_{\chi(2\,\mathrm{N}-1)} T_\varphi$, $V_2 = M_{\chi(2\,\mathrm{N})} T_\varphi$.



(4-1) $e = e_1 = \chi_{\{1\}}$ and the sequence $\{e_k\}_{k=1}^\infty$ is the canonical CONS of $\ell^2(\mathrm{N})$.

(4-2) $ONS(e,V)$ is complete.
(6) $M = B(\ell^2(\mathrm{N}))$.
(7) $A_\varphi(\eta)(k) = \eta(2k-1) + \eta(2k)$.



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